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Slinkies and the S-function content of certain generating functions

R C King, B G Wybourne and M Yang§

† Department of Mathematics, University of Southampton, Southampton, SO9 5NH, UK
‡ Department of Physics, University of Canterbury, Christchurch 1, New Zealand
§ Department of Mathematics, University of California at San Diego, La Jolla, 92093, USA

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Abstract. A variety of methods are described for expanding particular generating functions in terms of their S-function content. Plethysms are used to establish conjugacy relations between S-function series and the task of evaluating certain products of classical S-function series introduced by Littlewood is then completed. Slinky diagrams are introduced to represent non-standard S-functions and their modification to give standard S-functions. The role of slinkies in determining the S-function content of other generating functions is then explained and exemplified.

1. Introduction

The S-function is a special type of symmetric function that is closely linked to the character theory of the unitary and symmetric groups (Littlewood 1950, Wybourne 1970, Macdonald 1979). Infinite series of S-functions play a key role in the calculation of many properties of Lie groups such as in the evaluation of Kronecker products and branching rules (King 1975, King *et al* 1981, Black *et al* 1983, Black and Wybourne 1983). Similar series also arise in various aspects of non-compact groups (Rowe *et al* 1985, King and Wybourne 1985) and in string spectra (Farmer *et al* 1988).

It is possible both to construct generating functions for particular S-function series and, conversely, to determine the S-function content of particular generating functions by a variety of methods (Littlewood 1950, Knuth 1970, Stanley 1971, Bender and Knuth 1972, McConnell and Newell 1973, Burge 1974, Macdonald 1979, Josefiak and Weyman 1985, Yang and Wybourne 1986, Lascoux and Pragacz 1988).

This literature is dominated by the consideration of certain classical S-function series introduced by Littlewood (1950). Some of these are related to one another by means of conjugacy relations and the substitutional operation of plethysm. These aspects of the subject are discussed in § 3 with a view to setting the stage for the determination in § 4 of the S-function content of a particular set of products of the classical S-function series. Nothing other than a special case of the Littlewood-Richardson rule (Littlewood 1950, Macdonald 1979) is required to complete this exercise.

One method (Littlewood 1950, McConnell and Newell 1973) of determining the S-function content of certain generating functions first yields an infinite series of non-standard S-functions. These must then be converted into standard S-functions by the use of well known S-function modification rules (Murnaghan 1938, Littlewood

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1950). This approach has been exploited already to determine the S-function content of a number of new generating functions that produce infinite series of S functions (Yang and Wybourne 1986). In this paper we show how slinky diagrams (Chen *et al* 1984) may be used in this context to represent both non-standard S-functions and their subsequent modification to produce standard S-functions. The slinkies are defined in § 2 and exploited in § 5 in the determination of the S-function content of a number of new generating functions.

2. Slinkies and S-function modification rules

S-functions may be defined in various ways. For our purposes it is convenient to define (Macdonald 1979) the S-function, $s_\lambda(\boldsymbol{x})$, labelled by the partition $\lambda = (\lambda_1\lambda_2\ldots\lambda_p)$, with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p > 0$ and $\lambda_1 \in \mathbb{N}$ for $i = 1, 2, \ldots, p$, as the ratio of two $N \times N$ determinants:

$$\{\lambda\} = \{\lambda\}(\boldsymbol{x}) = s_\lambda(\boldsymbol{x}) = |x_i^{\lambda_j+N-j}|/|x_i^{N-j}| \tag{2.1}$$

where $(\boldsymbol{x}) = (x_1x_2\ldots x_N)$, with $N \geqslant p$, is a sequence of indeterminates whose presence is not always explicitly indicated and whose number, N, may usually be taken to be unbounded. The *length*, l_λ, of λ is the number of its non-vanishing parts, i.e. $l_\lambda = p$ and $\lambda_i = 0$ for $i > p$, and the *weight*, ω_λ, of λ is the sum of its parts, i.e. $\omega_\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_p$.

The link between S-functions and the character theory of groups is such that, if λ is a partition with $l_\lambda \leqslant N$ and the eigenvalues of a group element, g, of U_N are given by $x_j = \exp(i\phi_j)$ for $j = 1, 2, \ldots, N$ then the S-function $\{\lambda\} = \{\lambda_1\lambda_2\ldots\lambda_N\} = s_\lambda(\boldsymbol{x}) = s_\lambda(\exp(i\phi_1)\exp(i\phi_2)\ldots\exp(i\phi_N))$ is nothing other than the character of g in the irreducible representation of U_N conventionally denoted by $\{\lambda\}$ (Littlewood 1950).

To each partition $\lambda = (\lambda_1\lambda_2\ldots\lambda_p)$ of weight ω_λ and length $l_\lambda = p$ there corresponds a Young diagram or Ferrers diagram F^λ consisting of ω_λ boxes, nodes or circles arranged in l_λ left-adjusted rows of lengths $\lambda_1, \lambda_2, \ldots, \lambda_p$. For example, if $(\lambda) = (4333) = (43^3)$ then

$$F^\lambda = F^{4333} = \begin{array}{cccc} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \\ \circ & \circ & \circ & \\ \circ & \circ & \circ & \end{array}$$

Such a diagram is said to be *regular* by virtue of the fact that the row lengths are non-increasing from top to bottom. To each partition λ there corresponds a *conjugate* partition λ' defined in such a way that $F^{\lambda'}$ is obtained from F^λ by interchanging rows and columns. In our example $\lambda = (43^3)$ and $\lambda' = (4^31)$. An alternative specification of a partition is provided by the Frobenius notation (Littlewood 1950, Macdonald 1979) whereby one writes

$$\lambda = \begin{pmatrix} a_1a_2\ldots a_r \\ b_1b_2\ldots b_r \end{pmatrix}$$

where $a_i = \lambda_i - i$ and $b_i = \lambda_i' - i$ for $i = 1, 2, \ldots, r$ and r is the number of circles on the main diagonal of F^λ, so that $a_1 > a_2 > \ldots > a_r \geqslant 0$ and $b_1 > b_2 > \ldots > b_r \geqslant 0$. In our example we have $\lambda = \binom{310}{321}$.

Young diagrams offer the opportunity of giving an alternative, combinatorial, definition (Stanley 1971) of the S-function $\{\lambda\}$, namely

$$\{\lambda\}(\boldsymbol{x}) = \sum_a t_a^\lambda(\boldsymbol{x}) \tag{2.2}$$

where the summation is to be carried out over all distinct standard Young tableaux of shape F^λ. Such a standard Young tableau, indexed by a, is obtained by replacing each circle of F^λ by an entry taken from the set $\{1, 2, \ldots, N\}$ arranged so as to be weakly increasing across rows and strictly increasing down columns. If the entries in the tableau are $i_1, i_2, \ldots, i_{\omega_\lambda}$ then $t_a^\lambda(\boldsymbol{x}) = x_{i_1} x_{i_2} \ldots x_{i_{\omega_\lambda}}$.

More general S-functions $\{\lambda\}$ may, however, be defined by means of (2.1) with λ now any sequence $(\lambda_1\lambda_2\ldots\lambda_N)$, with $\lambda_i \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ for $i = 1, 2, \ldots, N$. Such an S-function is said to be *standard* if and only if λ is a partition; otherwise it is said to be *non-standard*. In the non-standard case the corresponding Young diagram F^λ, consisting of ω_λ boxes, nodes or circles arranged in left-adjusted rows of lengths λ_1, $\lambda_2, \ldots, \lambda_N$, is clearly not regular. For example, the Young diagram corresponding to the non-standard S-function $\{\lambda\} = \{60531070\}$ takes the form

$$F^\lambda = F^{60531070} = \begin{array}{lllllll} \circ & \circ & \circ & \circ & \circ & \circ & \\ \bullet & & & & & & \\ \circ & \circ & \circ & \circ & \circ & & \\ \circ & \circ & \circ & & & & \\ \circ & & & & & & \\ \bullet & & & & & & \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \bullet & & & & & & \end{array}$$

where it has been convenient to signify empty rows by means of a dot in the first column.

Each non-standard S-function is either identically zero or may be converted to standard form by suitably interchanging the columns of the determinant in the numerator of (2.1). This observation leads to the modification rules (Littlewood 1950):

$$\{\lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_N\} = -\{\lambda_1, \ldots, \lambda_{i+1} - 1, \lambda_i + 1, \ldots, \lambda_N\} \tag{2.3a}$$

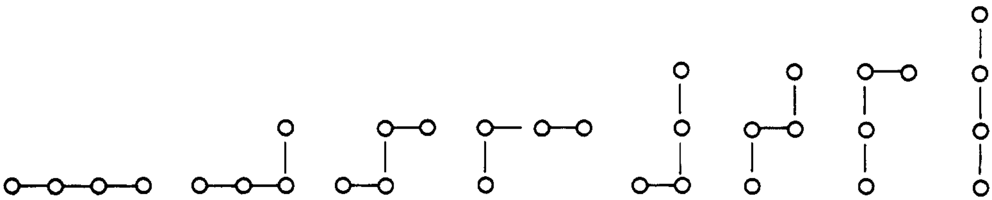
$$\{\lambda\} = 0 \qquad \text{if } \lambda_{i+1} = \lambda_i + 1. \tag{2.3b}$$

The application of (2.3a) is to be repeated until either (2.3b) indicates that the S-function is zero or a signed standard S-function is obtained.

While the above simple rules may be readily applied to a given non-standard S-function they do not offer much insight into the handling of infinite series of non-standard S-functions. Here our irregular Young diagrams have a role to play in conjunction with the notion of a *slinky* (Chen *et al* 1987).

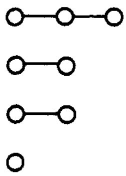
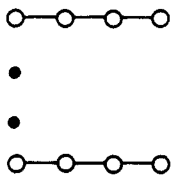
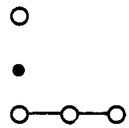
A slinky of length q is a diagram of q circles joined by $q-1$ links (Chen *et al* 1984). A slinky can be folded so as to take the shape of a continuous boundary strip of a regular Young diagram, with each of its links either horizontal or vertical and its circles forming part of the boundary of such a diagram. The *sign* of such a slinky is defined to be $(-1)^{r-1}$ where r is the number of rows occupied by the circles of the slinky, so that $r-1$ is the number of vertical links of the slinky.

By way of example a slinky of length 4 may be folded into the following eight standard continuous boundary strip shapes:



The signs are given by 1, −1, −1, −1, 1, 1, 1 and −1, respectively. It is to be noted that the *starting point* of each slinky is taken to be the position of the leftmost circle in the lowest row, and that from this position each slinky extends to the right or upwards in a sequence of right-angled folds.

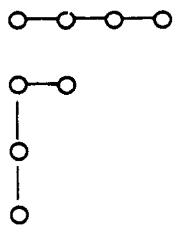
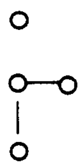
The Young diagram corresponding to any S-function $\{\lambda\}=\{\lambda_1\lambda_2\ldots\lambda_N\}$, whether or not it is standard, can be thought of as consisting of N horizontal slinkies. For $i=1, 2, \ldots, N$ the ith slinky has length λ_i and starts at the point $(i, 1)$ in the first column of the ith row, the case $\lambda_i=0$ being signified by a dot at $(i, 1)$. For example

$\{3221\}$:  $\{4004\}$:  $\{103\}$: 

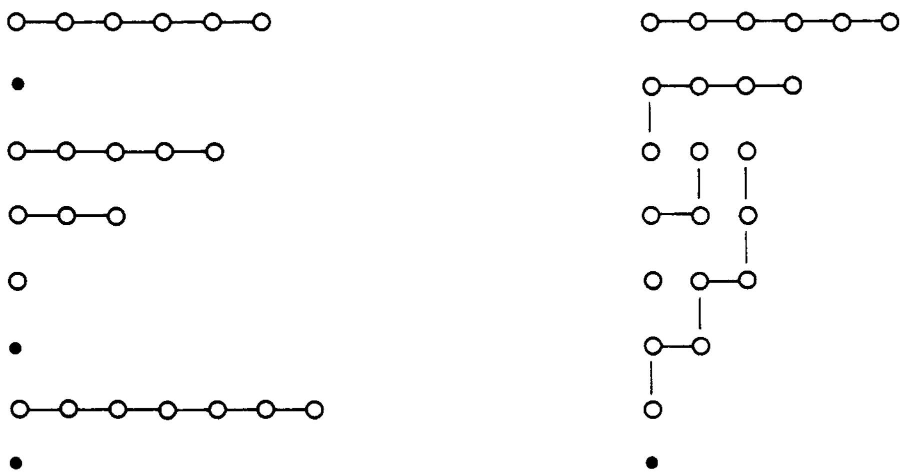
The modification rules (2.3) for non-standard S-functions can then be implemented in terms of folding operations on the slinkies constituting the Young diagrams as follows.

Draw the slinky diagram corresponding to the non-standard S-function $\{\lambda_1\lambda_2\ldots\lambda_N\}$. Then, successively for $i=1, 2, 3, \ldots, N$, while holding the starting positions of the slinkies fixed, fold (if necessary) the ith slinky of length λ_i into the shape of the unique standard continuous boundary strip such that the first i rows of the resulting diagram constitute a regular Young diagram. If this is not possible by virtue of the fact that, after folding, the resulting diagram is not regular then $\{\lambda\}=0$. Otherwise we obtain, after folding the last slinky, the regular Young diagram corresponding to some standard S-function, say $\{\mu\}$. The final result is then $\{\lambda\}=(-1)^v\{\mu\}$ where v is the total number of vertical links in the diagram.

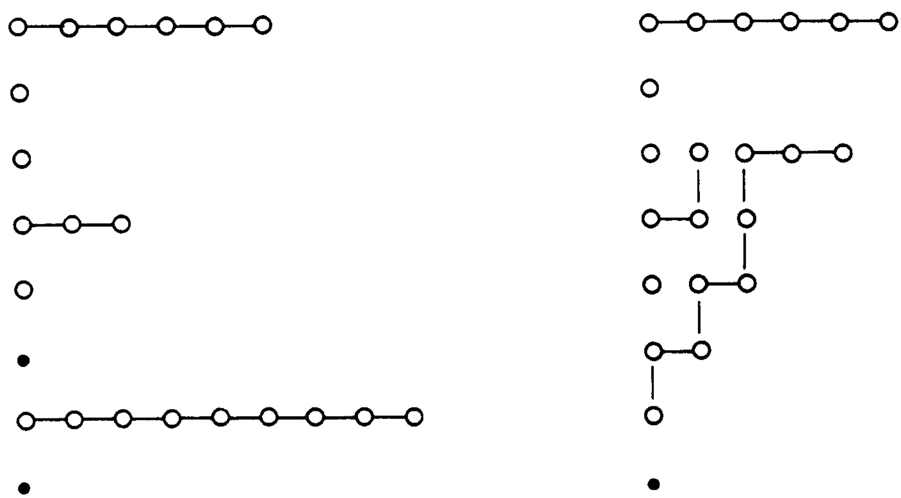
Clearly no modification is required in the case $\{3221\}$, whilst our procedure gives for $\{4004\}$ and $\{103\}$ the folded slinky diagrams

$\{4004\}$:  $\{103\}$: 

corresponding to the modification rules $\{4004\} = -\{4031\} = \{4211\}$ and $\{103\} = -\{121\} = 0$ which follow from the application of (2.3). The following diagrams illustrate the modification of the non-standard S-function $\{\lambda\} = \{60531070\}$ to give $\{64333210\}$:



As another example, $\{\lambda\} = \{61131090\} = 0$ as a consequence of the irregularity that appears when folding the fourth slinky of length 3 in the following diagrams (subsequent foldings are irrelevant):



We shall later illustrate the application of slinky techniques to some specific generating functions that give rise to infinite series of non-standard S-functions. Before doing so it is convenient to discuss some relationships between different series and then to evaluate a number of expansions explicitly.

3. Substitutions, plethysms and infinite series

In this paper we restrict attention only to those S-function series which are both *integral* and *invertible*, in that they take the general form

$$S(x)=\sum_{\lambda} c_\lambda\{\lambda\}(x) \qquad \text{with}\begin{cases} c_0=1 & \\ c_\lambda \in \mathbb{Z} & \text{for } \lambda \neq 0.\end{cases} \tag{3.1}$$

The summation is to be taken over all partitions λ including the partition 0 for which $\{0\}=1$. The number N of the indeterminates $x_1, x_2, \ldots, x_N$ is a parameter which can assume any positive integer value but which can be thought of in the general case as being unbounded. Many of the S-function series encountered here are *infinite*, in the sense that an infinite number of the coefficients c_λ are non-vanishing. The S-function series S is said to be *multiplicity free* if, for every non-zero c_λ, we have $c_\lambda=\pm 1$.

Associated with every series S of the form (3.1) there is a *conjugate* series $S=\Sigma_\lambda c_\lambda\{\lambda'\}$ where λ' is the partition conjugate to λ, and an *inverse* series S^{-1} such that

$$S\cdot S^{-1}=S^{-1}\cdot S=\{0\}=1. \tag{3.2}$$

The conjugate of the inverse series S^{-1} is defined to be the *adjoint* series $S^{\dagger}$ where $S^{\dagger}=(S^{-1})'=(S')^{-1}$. The four series S, S', S^{-1} and $S^{\dagger}$ are said to form a *family*. For example, the four series L, M, P and Q (Yang and Wybourne 1986) form the L family with $L'=P$, $L^{-1}=M$ and $L^{\dagger}=Q$. If $S=S'$ the series S is said to be *self-conjugate*, while if $S=S^{\dagger}$ it is *self-adjoint*.

As has been emphasised elsewhere (Yang and Wybourne 1986), almost all the classical S-function series (Littlewood 1950, King 1975) can be related by means of the substitutional operation of *plethysm* to the S-function series

$$L(x)=\prod_i (1-x_i). \tag{3.3}$$

It follows from (2.2) that, with the identification

$$\{\lambda\}(x)=\sum_a t_a^\lambda(x)=\sum_a y_a=\{1\}(y) \tag{3.4}$$

the corresponding plethysm $\{\lambda\}\otimes L$ takes the form

$$\{\lambda\}\otimes L(x)=L(y)=\prod_a (1-t_a^\lambda(x)). \tag{3.5}$$

In particular we have the following table of generating functions for the various families of classical S-function series (Littlewood 1950, King 1975, Yang and Wybourne 1986):

S-function series $S(x)$	S	S'	S^{-1}	$S^{\dagger}$
$\{1\}\otimes L(x)=L(x)=\prod_i(1-x_i)$	L	P	M	Q
$\{2\}\otimes L(x)=C(x)=\prod_{i\leqslant j}(1-x_ix_j)$	C	A	D	B
$\{1^2\}\otimes L(x)=A(x)=\prod_{i<j}(1-x_ix_j)$	A	C	B	D
$(\{1^2\}+\{1\})\otimes L(x)=E(x)=\prod_k(1-x_k)\prod_{i<j}(1-x_ix_j)$	E	E	F	F
$(\{1^2\}-\{1\})\otimes L(x)=G(x)=\prod_k(1+x_k)\prod_{i<j}(1-x_ix_j)$	G	G	H	H
$(\{2\}-\{1^2\})\otimes L(x)=V(x)=\prod_i(1-x_i^2)$	V	W	W	V

The algebra of plethysm (Littlewood 1950) and the properties of the L family of S-function series are such that

$$(\{\lambda\}\otimes L)' = \begin{cases} \{\lambda'\}\otimes L & \text{if } \omega_\lambda \text{ is even} \\ \{\lambda'\}\otimes L' & \text{if } \omega_\lambda \text{ is odd} \end{cases} \tag{3.6a}$$

$$(\{\lambda\}\otimes L)^{-1} = \{\lambda\}\otimes L^{-1} = (-\{\lambda\})\otimes L \tag{3.6b}$$

$$(\{\lambda\}+\{\mu\})\otimes L = (\{\lambda\}\otimes L)\cdot(\{\mu\}\otimes L). \tag{3.6c}$$

All of these formulae may be applied to the other members of the L family. It is only necessary to replace L by M, P and Q as appropriate.

A further link between the classical S-function series is provided by the transformation $x_j \to qx_j$, for all j, applied to any S-function $\{\lambda\}(\boldsymbol{x})$ or to any S-function series $S(\boldsymbol{x}) = \sum_\lambda c_l\{\lambda\}(\boldsymbol{x})$. It follows immediately from (2.2) that

$$\{\lambda\}(q\boldsymbol{x}) = q^{\omega_\lambda}\{\lambda\}(\boldsymbol{x}) \qquad S(q\boldsymbol{x}) = \sum_\lambda q^{\omega_\lambda} c_\lambda \{\lambda\}(\boldsymbol{x}). \tag{3.7}$$

It is then a trivial matter to see that, in the notation of Yang and Wybourne (1986), $Q(\boldsymbol{x}) = L(-\boldsymbol{x})$, $P(\boldsymbol{x}) = M(-\boldsymbol{x})$, $G(\boldsymbol{x}) = E(-\boldsymbol{x})$ and $H(\boldsymbol{x}) = F(-\boldsymbol{x})$, whilst for $S = A$, B, C, D, V and W we have $S(-\boldsymbol{x}) = S(\boldsymbol{x})$ and $S(\mathrm{i}\boldsymbol{x}) = S^{+}(\boldsymbol{x})$ where $\mathrm{i}^2 = -1$.

A further more general type of substitution may be made. Let

$$S_1(\boldsymbol{x}) = \prod_i (1 - f(x_i)) = \prod_{k,i} (1 - q_k x_i) \tag{3.8a}$$

where $f(x) = \Sigma_{k=1}^m a_k x^k$ is any polynomial in x of degree m, and the coefficients q_k are the reciprocals of the m roots of $(1 - f(x))$. Then we can define

$$S_2(\boldsymbol{x}) = \prod_{i\leqslant j} (1 - f(x_i x_j)) \tag{3.8b}$$

$$S_{11}(\boldsymbol{x}) = \prod_{i<j} (1 - f(x_i x_j)) \tag{3.8c}$$

and, more generally, in the notation of (3.4),

$$S_\lambda(\boldsymbol{x}) = \prod_a (1 - f(t_a^\lambda(\boldsymbol{x}))). \tag{3.9}$$

The conjugate $S'_\lambda(\boldsymbol{x})$ of $S_\lambda(\boldsymbol{x})$ may be evaluated, for example, by exploiting the Cauchy formula (Littlewood 1950, Stanley 1971, Macdonald 1979, Lascoux and Pragacz 1988):

$$\prod_{k,a} (1 - q_k y_a)^{-1} = \sum_\mu s_\mu(\boldsymbol{q}) s_\mu(\boldsymbol{y}) \tag{3.10}$$

and the related identity

$$\prod_{k,a} (1 - q_k y_a) = \sum_\mu (-1)^{\omega_\mu} s_{\mu'}(\boldsymbol{q}) s_\mu(\boldsymbol{y}). \tag{3.11}$$

Since (3.9) can be expanded in the form

$$\begin{aligned} S_\lambda(\boldsymbol{x}) &= \prod_a (1 - f(t_a^\lambda)) = \prod_{k,a} (1 - q_k y_a) = \sum_\mu (-1)^{\omega_\mu} s_{\mu'}(\boldsymbol{q}) s_\mu(\boldsymbol{y}) \\ &= \sum_\mu (-1)^{\omega_\mu} s_{\mu'}(\boldsymbol{q}) \{\lambda\}\otimes\{\mu\}(\boldsymbol{x}) \end{aligned} \tag{3.12}$$

and (Littlewood 1950)

$$(\{\lambda\}\otimes\{\mu\})' = \begin{cases} \{\lambda'\}\otimes\{\mu\} & \text{if } \omega_\lambda \text{ is even} \\ \{\lambda'\}\otimes\{\mu'\} & \text{if } \omega_\lambda \text{ is odd} \end{cases} \tag{3.13}$$

it follows that

$$S'_\lambda(x)=\begin{cases}\sum\limits_{\mu}(-1)^{\omega_\mu}s_{\mu'}(q)\{\lambda'\}\otimes\{\mu\}(x)=\sum\limits_{\omega}(-1)^{\omega_\mu}s_{\mu'}(q)s_\mu(z)\\ \sum\limits_{\mu}(-1)^{\omega_\mu}s_{\mu'}(q)\{\lambda'\}\otimes\{\mu'\}(x)=\sum\limits_{\mu}(-1)^{\omega_\mu}s_{\mu'}(q)s_{\mu'}(z)\end{cases}$$
$$=\begin{cases}\prod\limits_{k,b}(1-q_kz_b)=\prod\limits_{b}(1-f(t_b^{\lambda'}(x)))=S_{\lambda'}(x) & \text{if } \omega_\lambda \text{ is even}\\ \prod\limits_{k,b}(1+q_kz_b)^{-1}=\prod\limits_{b}(1-f(-t_b^{\lambda'}(x)))^{-1}=S_{\lambda'}^{-1}(-x) & \text{if } \omega_\lambda \text{ is odd}\end{cases} \tag{3.14}$$

where $z_b=t_b^{\lambda'}(x)$ indicates a contribution to $S_{\lambda'}(z)$ arising from a standard Young tableaux of shape $F^{\lambda'}$.

Taking $f(x)=x$ in the specification (3.9) of $S_\lambda(x)$ merely allows us to confirm the validity of the conjugacy relations $L'=P$ and $C'=A$ by using (3.14) with $\lambda=1$ and $\lambda=2$, respectively. Similarly the result $V'=W$ follows from (3.14) in the case $f(x)=x^2$ and $\lambda=1$. More generally, for all $f(x)$, it follows from (3.14) that

$$S'_1(x)=S_1^{-1}(-x) \qquad S_1^{\dagger}(x)=S_1(-x) \tag{3.15a}$$

and

$$S'_{11}(x)=S_2(x) \qquad S_{11}^{\dagger}(x)=S_2^{-1}(x). \tag{3.15b}$$

Thus, if

$$S_1(x)=\prod_i(1+a_1x_1+a_2x_1^2+\ldots+a_mx_i^m) \tag{3.16a}$$

then

$$S'_1(x)=\prod_i(1-a_1x_i+a_2x_i^2-\ldots+(-1)^m a_mx_i^m)^{-1} \tag{3.16b}$$

$$S_1^{-1}(x)=\prod_i(1+a_1x_i+a_2x_i^2+\ldots+a_mx_i^m)^{-1} \tag{3.16c}$$

$$S_1^{\dagger}(x)=\prod_i(1-a_1x_i+a_2x_i^2-\ldots+(-1)^m a_mx_i^m). \tag{3.16d}$$

It is the evaluation of the S-function series of type (3.16) which will be undertaken in § 5 as an illustration of the use of slinkies, but first we turn to some products of the classical S-function series.

4. Products of S-function series

We are now in a position to complete the task of evaluating products of two types, namely the 16 products

$$\prod_i(1\pm x_i)^{\pm1}\prod_j(1\pm x_j)^{\pm1} \tag{4.1}$$

and the 32 products

$$\prod_k(1\pm x_k)^{\pm1}\prod_{\substack{i<j\\ i\leqslant j}}(1\pm x_ix_j)^{\pm1}. \tag{4.2}$$

This task was begun (Littlewood 1950, King 1975) and nearly completed (Yang and Wybourne 1986, Lasoux and Pragacz 1988) elsewhere. We both unify and generalise the results by exploiting (3.7).

The various conjugacy relations given in § 3 imply that the problem reduces to the evaluation of the following products:

$$Q(px)Q(qx)=\prod_i (1+px_i)\prod_j (1+qx_j) \tag{4.3a}$$

$$Q(px)M(qx)=\prod_i (1+px_i)\prod_j (1-qx_j)^{-1} \tag{4.3b}$$

for (4.1), and

$$Q(px)A(\mathrm{i}qx)=\prod_k (1+px_k)\prod_{i<j} (1+q^2x_ix_j) \tag{4.4a}$$

$$Q(px)B(qx)=\prod_k (1+px_k)\prod_{i<j} (1-q^2x_ix_j)^{-1} \tag{4.4b}$$

$$Q(px)C(\mathrm{i}qx)=\prod_k (1+px_k)\prod_{i\leqslant j} (1+q^2x_ix_j) \tag{4.4c}$$

$$Q(px)D(qx)=\prod_k (1+px_k)\prod_{i\leqslant j} (1-q^2x_ix_j)^{-1}. \tag{4.4d}$$

in the case of (4.2).

Probably the simplest way to evaluate (4.3a), or any generalisation of it involving further factors of the type $Q(rx)$, is to make use of (3.11). This immediately gives

$$\prod_k Q(q_kx)=\sum_\lambda \{\lambda\}(\boldsymbol{q})\{\lambda\}'(x) \tag{4.5}$$

so that in our special case

$$\begin{aligned} Q(px)Q(qx)&=\sum_\lambda \{\lambda\}(p,q)\{\lambda\}'(x)\\ &=\sum_{s,t\geqslant 0}(pq)^t(p^s+p^{s-1}q+\ldots+q^s)\{s+t,t\}'(x). \end{aligned} \tag{4.6}$$

In deriving this result (2.2) has been used, along with the recognition that in the case of two indeterminates, p and q, the only relevant standard Young tableaux are typically of the form

$$\begin{matrix} 1&1&1&1&1&1&1&1&1&1&2&2&2&2\\ 2&2&2&2&2&2&2 \end{matrix}$$

and thus of shape $F^{(s+t,t)}$ for some s and t, with $s\geqslant 0$ and $t\geqslant 0$.

Setting, in turn, $p=q=1$, $p=-q=1$ and $p=q=-1$ in (4.6) gives (Black *et al* 1983, Black and Wybourne 1983)

$$Q(x)Q(x)=\sum_{s,t\geqslant 0}(s+1)\{s+t,t\}'(x) \tag{4.7a}$$

$$Q(x)L(x)=V(x)=\sum_{s,t\geqslant 0}(-1)^s\{s+2t,s\}'(x) \tag{4.7b}$$

and

$$L(x)L(x)=\sum_{s,t\geqslant 0}(-1)^s(s+1)\{s+t,t\}'(x). \tag{4.7c}$$

The conjugates of these expansions yield $M(x)M(x)$, $M(x)P(x)=W(x)$ and $P(x)P(x)$.

In the case of (4.3b) the usual Littlewood–Richardson rule (Littlewood 1950, Macdonald 1979) for multiplying S-functions may be used to show that

$$Q(px)M(qx)=\prod_i(1+px_i)\prod_j(1-qx_j)^{-1}=\sum_{m,n\geqslant 0}p^mq^n\{1^m\}(x)\{n\}(x)$$

$$=1+\sum_{a,b\geqslant 0}p^bq^a(p+q)\begin{pmatrix}a\\b\end{pmatrix}(x). \tag{4.8}$$

The special cases $p=\pm1$ and $q=\pm1$ then yield, in the notation of Yang and Wybourne (1986), the familiar results $Q(x)M(x)=S(x)$, $Q(x)P(x)=L(x)M(x)=1$ and $L(x)P(x)=R(x)$. This completes the evaluation of the products (4.1) by means of (4.3).

All that is required to effect the expansions of (4.4) is to note first that the constituent series are given by (Littlewood 1950)

$$Q(px)=\prod_k(1+px_k)=\sum_m p^m\{1^m\}(x) \tag{4.9a}$$

$$A(\mathrm{i}qx)=\prod_{i<j}(1+q^2x_ix_j)=\sum_\alpha q^{\omega_\alpha}\{\alpha\}(x) \tag{4.9b}$$

$$B(qx)=\prod_{i<j}(1-q^2x_ix_j)^{-1}=\sum_\beta q^{\omega_\beta}\{\beta\}(x) \tag{4.9c}$$

$$C(\mathrm{i}qx)=\prod_{i\leqslant j}(1+q^2x_ix_j)=\sum_\gamma q^{\omega_\gamma}\{\gamma\}(x) \tag{4.9d}$$

$$D(qx)=\prod_{i\leqslant j}(1-q^2x_ix_j)^{-1}=\sum_\delta q^{\omega_\delta}\{\delta\}(x) \tag{4.9e}$$

where

$$\alpha=\gamma'=\begin{pmatrix}a_1 & a_2 & \dots\\ a_1+1 & a_2+1 & \dots\end{pmatrix}$$

and $\beta'=\delta=(2\lambda_1,2\lambda_2,\dots)$, and then to evaluate the products of all the terms in the last four expressions with $\{1^m\}$. These products are given by the special case of the Littlewood–Richardson rule known as Pieri's rule: $\{1^m\}\{\lambda\}=\sum_\mu\{\mu\}$, where for each partition λ the sum is over all partitions μ such that F^μ is a regular diagram obtained by adding m circles to F^λ with at most one added to any given row. It is then straightforward but tedious to obtain the following results:

$$Q(px)A(\mathrm{i}qx)=\sum\left\{\prod_{i=1}^r\mathrm{mult}\begin{pmatrix}a_i\\b_i\end{pmatrix}\right\}\begin{pmatrix}a_1a_2\dots a_r\\b_1b_2\dots b_r\end{pmatrix}(x) \tag{4.10a}$$

where

$$\mathrm{mult}\begin{pmatrix}a_i\\b_i\end{pmatrix}=p^{b_i-a_i-1}q^{2a_i}\begin{cases}0 & \text{if } a_i>b_i \quad (a_r>b_r \quad \text{if } i=r)\\ p^2 & \text{if } a_i=b_i \quad (a_r=b_r \quad \text{if } i=r)\\ p^2+q^2 & \text{if } b_{i+1}<a_i<b_i \quad (0\leqslant a_r<b_r \text{ if } i=r)\\ q^2 & \text{if } a_i=b_{i+1}\\ 0 & \text{if } a_i<b_{i+1}\end{cases} \tag{4.10b}$$

and

$$Q(px)B(qx)=\sum_\lambda\left\{\prod_i\mathrm{mult}(\lambda_i')\right\}\{\lambda\}(x) \tag{4.11a}$$

where

$$\operatorname{mult}(\lambda'_1)=\begin{cases} q^{\lambda'_i}+q^{\lambda'_i-2}p^2+\ldots+q^{\lambda'_{i+1}}p^{\lambda'_i-\lambda'_{i+1}} & \text{for } \lambda'_i\equiv\lambda'_{i+1}\equiv 0(\operatorname{mod} 2)\\ q^{\lambda'_i}+q^{\lambda'_i-2}p^2+\ldots+q^{\lambda'_{i+1}+1}p^{\lambda'_i-\lambda'_{i+1}-1} & \text{for } \lambda'_1\equiv 0(\operatorname{mod} 2) \text{ and } \lambda'_{i+1}\equiv 1(\operatorname{mod} 2)\\ q^{\lambda'_i-1}p+q^{\lambda'_i-3}p^3+\ldots+q^{\lambda'_{i+1}}p^{\lambda'_i-\lambda_{i+1}} & \text{for } \lambda'_i\equiv 1(\operatorname{mod} 2) \text{ and } \lambda'_{i+1}\equiv 0(\operatorname{mod} 2)\\ q^{\lambda'_i-1}p+q^{\lambda'_i-3}p^3+\ldots+q^{\lambda'_{i+1}+1}p^{\lambda'_i-\lambda'_{i+1}-1} & \text{for } \lambda'_1\equiv\lambda'_{i+1}\equiv 1(\operatorname{mod} 2); \end{cases} \tag{4.11b}$$

and

$$Q(px)C(\mathrm{i}qx)=\sum\left\{\prod_{i=1}^{r}\operatorname{mult}\binom{a_i}{b_i}\right\}\begin{pmatrix}a_1a_2\ldots a_r\\ b_1b_2\ldots b_r\end{pmatrix}(x) \tag{4.12a}$$

where

$$\operatorname{mult}\binom{a_i}{b_i}=p^{b_i-a_i+1}q^{2a_i-2}\begin{cases} 0 & \text{if } a_i>b_i+2 & (a_r>b_r+2 & \text{if } i=r)\\ p^2 & \text{if } a_i=b_i+2 & (a_r=b_r+2 & \text{if } i=r)\\ p^2+q^2 & \text{if } b_{i+1}+2<a_i<b_i+2 & (1<a_r<b_r+2 & \text{if } i=r)\\ q^2 & \text{if } a_i=b_{i+1}+2 & (a_r=0,1 & \text{if } i=r)\\ 0 & \text{if } a_i<b_{i+1}+2 \end{cases} \tag{4.12b}$$

and finally

$$Q(px)D(qx)=\sum_{\lambda}\left\{\prod_{i}\operatorname{mult}(\lambda_i)\right\}\{\lambda\}(x) \tag{4.13a}$$

where

$$\operatorname{mul}(\lambda_i)=\begin{cases} q^{\lambda_i} & \text{if } \lambda_i\equiv 0(\operatorname{mod} 2)\\ q^{\lambda_i-1}p & \text{if } \lambda_i\equiv 1(\operatorname{mod} 2) \end{cases} \tag{4.13b}$$

so that

$$Q(px)D(qx)=\sum_{\lambda}p^{n_0}q^{\omega_\lambda-n_0}\{\lambda\}(x) \tag{4.13c}$$

where the summation is carried out over all partitions λ, and n_0 is the number of rows of F^λ of odd length.

Various special cases may be recovered from (4.10)-(4.13). For example, in the special case $p=1$ and $q=-i$ (4.10*b*) reduces to

$$\operatorname{mult}\binom{a_i}{b_i}=\begin{cases}(-1)^{a_i} & \text{if } a_i=b_i\\ 0 & \text{otherwise.}\end{cases}$$

so that

$$Q(x)A(x)=G(x)=\sum_{\varepsilon}(-1)^{(\omega_\varepsilon-r)/2}\{\varepsilon\}(x) \qquad \text{where } \varepsilon=\begin{pmatrix}a_1a_2\ldots a_r\\ a_1a_2\ldots a_r\end{pmatrix}. \tag{4.14}$$

Similarly the case $p=-1$ and $q=-i$ in (4.12*b*) gives

$$\operatorname{mult}\binom{a_i}{b_i}=\begin{cases}(-1)^{b_i-1} & \text{if } a_i=b_i\\ 0 & \text{otherwise}\end{cases}$$

so that

$$L(x)A(x)=E(x)=\sum_{\varepsilon}(-1)^{(\omega_\varepsilon+r)/2}\{\varepsilon\}(x) \qquad \text{where } \varepsilon=\begin{pmatrix}a_1a_2\ldots a_r\\ a_1a_2\ldots a_r\end{pmatrix}. \tag{4.15}$$

Other results of Yang and Wybourne (1986) may be recovered by setting $p=\pm 1$ and $q=\pm 1$.

Similarly the case $p=-1$ and $q=-i$ in (4.12b) gives

$$\operatorname{mult}\binom{a_i}{b_i}=\begin{cases}(-1)^{b_i} & \text{if } a_i=b_i+2\\ (-1)^{b_i+1} & \text{if } a_i=b_{i+1}+2\ (a_r=0 \text{ or } 1 \text{ if } i=r)\\ 0 & \text{otherwise}\end{cases}$$

leading to the result

$$L(\boldsymbol{x})C(\boldsymbol{x})=\sum_r\sum_{s=0}^{r}(-1)^{r-s+b_1+b_2+\ldots+b_r}\begin{pmatrix}a_1a_2\ldots a_r\\ b_1b_2\ldots b_r\end{pmatrix}(\boldsymbol{x}) \tag{4.16a}$$

with

$$a_i=\begin{cases}b_i+2 & \text{for } 1\leqslant i\leqslant s\leqslant r\\ b_{i+1}+2 & \text{for } s+1\leqslant i\leqslant r-1 \text{ if } s<r\\ 0 \text{ or } 1 & \text{for } i=r \text{ if } s<r.\end{cases} \tag{4.16b}$$

This is a slightly more explicit form of the result given by Lascoux and Pragacz (1988). Finally, their deliberate omission of a representative of a final family of products can be rectified by using (4.11) in the case $p=q=1$ to give

$$Q(\boldsymbol{x})B(\boldsymbol{x})=\sum\left\{\prod_i \operatorname{mult}(\lambda_i')\right\}\{\lambda\}(\boldsymbol{x}) \tag{4.17a}$$

where

$$\operatorname{mult}(\lambda_i')=\begin{cases}(\lambda_i'-\lambda_{i+1}'+2)/2 & \text{for } \lambda_i'\equiv\lambda_{i+1}'\equiv 0(\operatorname{mod} 2)\\ (\lambda_i'-\lambda_{i+1}'+1)/2 & \text{for } \lambda_i'\equiv 0(\operatorname{mod} 2) \text{ and } \lambda_{i+1}'\equiv 1(\operatorname{mod} 2)\\ (\lambda_i'-\lambda_{i+1}'+1)/2 & \text{for } \lambda_i'\equiv 1(\operatorname{mod} 2) \text{ and } \lambda_{i+1}'\equiv 0(\operatorname{mod} 2)\\ (\lambda_i'-\lambda_{i+1}')/2 & \text{for } \lambda_i'\equiv\lambda_{i+1}'\equiv 1(\operatorname{mod} 2).\end{cases} \tag{4.17b}$$

It should be pointed out that the conjugacy relations (3.15) imply that $QA=(MC)'$, $QB=(MD)'$, $QC=(MA)'$ and $QD=(MB)'$. To weight 8 we have from (4.16)

$$\begin{aligned}LC=&\{0\}-\{1\}+\{1^2\}-\{1^3\}+\{1^4\}-\{1^5\}+\{1^6\}-\{1^7\}+\{1^8\}\\ &-\{2\}+\{21\}-\{21^2\}+\{21^3\}-\{21^4\}+\{21^5\}-\{21^6\}+\{3\}-\{32\}\\ &+\{321\}-\{321^2\}+\{321^3\}-\{3^2\}+\{3^21\}-\{3^21^2\}-\{41\}+\{42\}\\ &-\{42^2\}+\{43\}-\{4^2\}+\{51^2\}-\{521\}+\ldots.\end{aligned} \tag{4.18}$$

and from the conjugate of (4.17)

$$\begin{aligned}MD=&\{0\}+\{1\}+2\{2\}+\{21\}+2\{2^2\}+\{2^21\}+2\{2^3\}+\{2^31\}+2\{2^4\}\\ &+2\{3\}+\{31\}+2\{32\}+\{321\}+2\{32^2\}+\{32^21\}+3\{4\}+2\{41\}\\ &+4\{42\}+2\{421\}+4\{42^2\}+2\{43\}+\{431\}+3\{4^2\}+3\{5\}+2\{51\}\\ &+4\{52\}+2\{521\}+2\{53\}+4\{6\}+3\{61\}+6\{62\}+4\{7\}+3\{71\}\\ &+5\{8\}+\ldots.\end{aligned} \tag{4.19}$$

5. Slinky diagrams and the S-function content of series

Bearing in mind the definition (2.1) of an S-function, the generating function

$$S_1(x)=\prod_i (1+a_i x_i+a_2 x_i^2+\ldots+a_m x^m) \tag{5.1}$$

may be expanded in terms of S-functions through a consideration of the product $S_1(x)|x_i^{N-j}|$. Multiplying the ith row of the Vandermonde determinant by the ith factor of $S_1(x)$ for $i=1, 2, \ldots, n$ immediately gives

$$S_1(x)=\sum_\mu \left\{\begin{matrix} 0 & 0 & \ldots & 0 \\ a_1 1 & a_1 1 & & a_1 1 \\ a_2 2 & a_2 2 & & a_2 2 \\ \vdots & \vdots & & \vdots \\ am & am & & am \end{matrix}\right\} \begin{matrix} \ldots \\ \ldots \\ \ldots\ (x) \\ \\ \ldots \end{matrix} \tag{5.2}$$

where the notation is intended to indicate that the summation is over all those S-functions $\{\mu\}(x)$ for which each part μ_i is taken from the set $\{0, 1, \ldots, m\}$ and carries with it a multiplicative factor a_{μ_i}, with $a_0=1$, for $i=1, 2, \ldots, N$. Thus

$$S_1(x)=\sum_\mu a(\mu)\{\mu\}(x) \qquad \text{with} \qquad a(\mu)=\prod_i a_{\mu_1} \qquad \text{where } \mu_i\in\{0, 1, \ldots, m\}. \tag{5.3}$$

Of course, in this expression μ is, in general, not a partition. Indeed the summation is over all μ such that F^μ consists of N horizontal slinkies of lengths $0, 1, \ldots,$ or m taken in any order with any number of repetitions. To standardise the expansion (5.2) it is only necessary to successively fold the slinkies of length $\mu_1, \mu_2, \ldots, \mu_N$ as described in § 2. This then yields an expansion of the required form:

$$S_1(x)=\sum_\lambda c(\lambda)\{\lambda\}(x) \qquad \text{with} \qquad m\geqslant\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_p>0;\ p\leqslant N. \tag{5.4}$$

In this expression

$$c(\lambda)=\sum_\mu (-1)^v a(\mu) \qquad \text{with} \qquad a(\mu)=\prod_{i=1}^N a_{\mu_i}=\prod_{k=1}^m a_k^{\alpha_k} \tag{5.5}$$

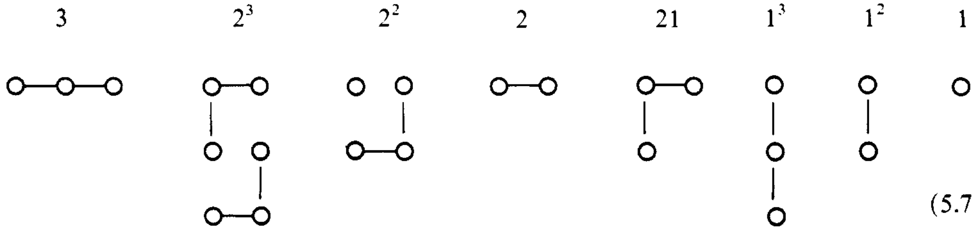
where the summation is over those μ such that the corresponding horizontal slinky diagram may be folded to give the slinky diagram of the partition λ, and v is the number of vertical folds. The exponents α_k serve to count the number of slinkies of length k.

It is apparent that, in the general case for which $N\to\infty$, the multiplicities $c(\lambda)$ can grow without limit. Nevertheless it is possible to construct complete results for low values of m.

Let us consider the specific case of $m=3$. We have

$$\prod_i (1+ax_i+bx_i^2+cx_i^3)=\sum_{p,q,r=0}^{\infty} g_{pqr}(a, b, c)\{3^p 2^q 1^r\}. \tag{5.6}$$

The lengths of the slinkies can only be 0, 1, 2 or 3. The list of all *elementary* regular combinations of folded slinkies is as follows:



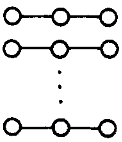
(5.7)

and the corresponding signed weight factors $(-1)^v a^\alpha b^\beta c^\gamma$ are

$$+c \qquad +c^2 \qquad -ac \qquad +b \qquad -c \qquad +c \qquad -b \qquad +a$$

The significance of the epithet *elementary* lies in the fact that all regular slinky diagrams with no more than three columns may be constructed using these particular regular slinky diagrams. In general, any one regular diagram may be constructed in more than one way from these building blocks.

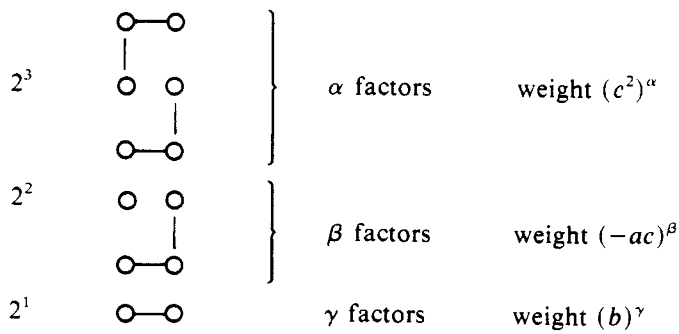
Consider the construction of the diagram corresponding to $\{3^p 2^q 1^r\}$. There is only one slinky diagram which will give rise to the term $\{3^p\}$, namely the diagram



composed of p copies of 3. The corresponding signed weighting factor P_p, of $\{3^p\}$ is given by

$$P_p = c^p. \tag{5.8}$$

For the term $\{2^q\}$ the situation is more complex. Using the elementary diagrams of (5.7) we can combine α copies of 2^3, β copies of 2^2 and γ copies of 2 to make up 2^q as shown schematically below:



where $q = 3\alpha + 2\beta + \gamma$. The various contributions to the signed weight factor $(c^2)^\alpha(-ac)^\beta(b)^\gamma$ have been indicated and there is an additional overall multiplicity of $\left[{}^{\alpha+\beta+\gamma}_{\alpha\ \beta\ \gamma}\right] = (\alpha+\beta+\gamma)!/\alpha!\beta!\gamma!$ arising from the fact that the $\alpha+\beta+\gamma$ factors may be arbitrarily ordered. The total multiplicity, Q_q, of the term $\{2^q\}$ is therefore given by

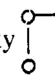
$$Q_q = \sum_{\substack{\alpha,\beta,\gamma \\ 3\alpha+2\beta+\gamma=q}} (-1)^\beta a^\beta b^\gamma c^{2\alpha+\beta} \begin{bmatrix} \alpha+\beta+\gamma \\ \alpha \quad \beta \quad \gamma \end{bmatrix}. \tag{5.9}$$

Similarly, the multiplity R_r of $\{1^r\}$ is given by

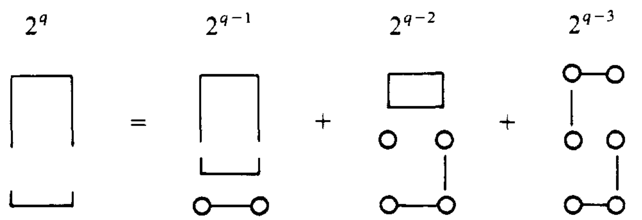
$$R_r = \sum_{\substack{\rho,\sigma,\tau \\ 3\rho+2\sigma+\tau=r}} (-1)^\sigma a^\tau b^\sigma c^\rho \begin{bmatrix} \rho+\sigma+\tau \\ \rho \quad \sigma \quad \tau \end{bmatrix}. \tag{5.10}$$

For the general term $\{3^p 2^q 1^r\}$ we then have

$$g_{pqr}(abc) = P_p(Q_q R_r - cQ_{q-1} R_{r-1}) \tag{5.11}$$

in which the second term is due to the presence of the slinky  of signed weight factor $-c$, which connects factors $\{2^{q-1}\}$ and $\{1^{r-1}\}$ to form $\{2^q 1^r\}$.

Using (5.8)-(5.10) in (5.11) gives an explicit expression for all the coefficients in the S-function series (5.6). In practice it is easier to use recurrence relations to determine both Q_q and R_r. These can be obtained through the use of multinomial identities or, better still, through a consideration of the structure of regular slinky diagrams. For example we have



(5.12)

which immediately yields the recurrence relation

$$Q_q = bQ_{q-1} - acQ_{q-2} + c^2 Q_{q-3} \qquad \text{for } q \geq 3 \qquad \text{with } Q_0 = 1,\ Q_1 = b,\ Q_2 = b^2 - ac. \tag{5.13}$$

Similarly

$$R_r = aR_{r-1} - bR_{r-2} + cR_{r-3} \qquad \text{for } r \geq 3 \qquad \text{with } R_0 = 1,\ R_1 = a,\ R_2 = a^2 - b. \tag{5.14}$$

For special values of a, b and c we obtain particularly simple results. Thus, for $a = b = c = 1$ we obtain the multiplicity-free expansion:

$$\begin{aligned}\prod_i (1 + x_i + x_i^2 + x_i^3) &= \sum_{p,q,r=0}^{\infty} g_{pqr}(1,1,1)\{3^p 2^q 1^r\} \\ &= \sum_{p,q,r=0}^{\infty} (\{3^p 2^{4q} 1^{4r}\} + \{3^p 2^{4q} 1^{4r+1}\} + \{3^p 2^{4q+1} 1^{4r}\} \\ &\quad - \{3^p 2^{4q+1} 1^{4r+2}\} - \{3^p 2^{4q+2} 1^{4r+1}\} - \{3^p 2^{4q+2} 1^{4r+2}\}).\end{aligned} \tag{5.15}$$

We note that, in this case, the left-hand side factorises as $\Pi_i\,(1+x_i)(1+x_i^2)=QV_+$ and hence the above result gives a compact account of the terms that arise in QV_+. We list below the terms of up to weight 8 as computed explicitly using SCHUR to form the product QV_+:

$$\{0\}+\{1\}+\{1^4\}+\{1^5\}+\{1^8\}+\{2\}-\{21^2\}+\{21^4\}-\{21^6\}-\{2^21\}-\{2^21^2\}+\{2^4\}+\{3\}+\{31\} \\ +\{31^4\}+\{31^5\}+\{32\}-\{321^2\}-\{32^21\}+\{3^2\}+\{3^21\}+\{3^22\}+\ldots \tag{5.16}$$

We can immediately deduce from (5.15) and (3.7) or (3.16d) that

$$\prod_i (1-x_i+x_i^2-x_i^3) \\ = \sum_{p,q,r=0}^{\infty} (-1)^p(\{3^p2^{4q}1^{4r}\}-\{3^p2^{4q}1^{4r+1}\}+\{3^p2^{4q+1}1^{4r}\} \\ -\{3^p2^{4q+1}1^{4r+2}\}+\{3^p2^{4q+2}1^{4r+1}\}-\{3^p2^{4q+2}1^{4r+2}\}) \tag{5.17}$$

and again by factorisation we find that the left-hand side is given by $\Pi_i\,(1-x_i)(1+x_i^2)=LV_+$. Up to terms of weight 8 we obtain

$$\{0\}-\{1\}+\{1^4\}-\{1^5\}+\{1^8\}+\{2\}-\{21^2\}+\{21^4\}-\{21^6\}+\{2^21\}-\{2^21^2\}+\{2^4\}-\{3\}+\{31\} \\ -\{31^4\}+\{31^5\}-\{32\}+\{321^2\}-\{32^21\}+\{3^2\}-\{3^21\}+\{3^22\}+\ldots. \tag{5.18}$$

The expansions of the inverse series $\Pi_i\,(1+x_i+x_i^2+x_i^3)^{-1}$ and $\Pi_i\,(1-x_i+x_i^2-x_i^3)^{-1}$ are then given, thanks to (3.15a) or (3.16b), by the conjugates of (5.17) and (5.15), respectively.

Other special cases of (5.6) have been dealt with elsewhere (Yang and Wybourne 1986) including the cases $(a=\pm1, b=c=0)$, $(b=\pm1, a=c=0)$, $(c=\pm1, a=b=0)$ and $(a=\pm1, b=1, c=0)$, whilst the special cases of (5.1) with $(a_i=0$ for $i=1,2,\ldots,m-1,\ a_m=\pm1)$ have also been discussed (Yang and Wybourne 1986, Lascoux and Pragacz 1988). These cases are all multiplicity free. We close with two examples involving multiplicities. First we consider the case of (5.6) with $a=b=-1$, $c=1$:

$$\Pi_i\,(1-x_i-x_i^2+x_i^3)=\sum_{p,q,r=0}^{\infty} g_{pqr}(-1,-1,1)\{3^p2^q1^r\} \tag{5.19}$$

where

$$g_{pqr}(-1,-1,1)=P_p(Q_qR_r-Q_{q-1}R_{r-1}) \tag{5.20}$$

with

$$P_p=1$$

$$Q_q=-Q_{q-1}+Q_{q-2}+Q_{q-3} \qquad \text{for } q\geqslant 3 \qquad \text{with } Q_0=1,\ Q_1=-1,\ Q_2=2$$

$$R_r=-R_{r-1}+R_{r-2}+R_{r-3} \qquad \text{for } r\geqslant 3 \qquad \text{with } R_0=1,\ R_1=-1,\ R_2=2.$$

Hence $Q_{2k}=-Q_{2k+1}=R_{2k}=-R_{2k+1}=k+1$, leading to the result

$$\prod_i (1-x_i-x_i^2+x_i^3) \\ = \sum_{p,q,r=0}^{\infty} ((q+r+1)\{3^p2^{2q}1^{2r}\}-(r+1)\{3^p2^{2q}1^{2r+1}\}-(q+1)\{3^p2^{2q+1}1^{2r}\}). \tag{5.21}$$

Finally the case $a=1$, $b=-1$, $c=0$ gives

$$\prod_i (1+x_i-x_i^2)=\sum_{p,q,r=0}^{\infty} g_{pqr}(1,-1,0)\{3^p2^q1^r\}=\sum_{q,r=0}^{\infty} Q_qR_r\{2^q1^r\} \tag{5.22}$$

where $Q_q=-Q_{q-1}$ with $Q_0=1$, and $R_r=R_{r-1}+R_{r-2}$ with $R_0=R_1=1$. Hence $Q_q=(-1)^q$ and $R_r=f_{r+1}$, the $(r+1)$th Fibonacci number. This gives

$$\prod_i (1+x_i-x_i^2)=\sum_{q,r=0}^{\infty}(-1)^q f_{r+1}\{2^q1^r\}. \tag{5.23}$$

Once more (3.16) may be exploited to write down a complete family of such expansions.

To conclude, we have shown how slinky diagrams can be used to determine the S-function content of generating functions that give rise to infinite series of S functions that may or may not be multiplicity free. We have also noted the role of substitutions and plethysms in the analysis of S-function series.

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